Independent Dominating Polynomial of Corona of Some Special Graphs

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Abstract

Consider a graph G with vertex set V. A set $S \subseteq V$ is independent (or stable) if no two vertices in it are adjacent. The set $S \subseteq V$ is a dominating set if every vertex in $V - S$ is adjacent to a vertex of S.

A subset S of V is an independent dominating set if it is independent and dominating in graph G. Independent domination number is the minimum size of an independent dominating set of G and is denoted by $\gamma_i(G)$. The independent dominating polynomial of a graph G of order n is the polynomial $D_i(G,x) = \sum_{j=\gamma_i(G)}^{n} d_i(G,j)x^j$, where $d_i(G,j)$ is the number of independent dominating sets of G of size j.

In this paper, the researcher determined the independent dominating polynomial of $K_n \circ G$, $K_{m,n} \circ G$ and $F_m \circ G$.

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1 INTRODUCTION

Independent domination problem is one of the interest in graph theory. The theory of independent domination was formalized by Berge and Ore in 1962 and the independent domination number were introduced by Cockayne and Hedetniemi.

Independent dominating sets which made up the independent dominating polynomials, arose in chessboard problems. In 1862, de Jaenisch posed the problem of finding the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens.

Graph polynomials are powerful and well-developed tools to express graph parameters. Usually, graph polynomials are compared to each other by adhoc means allowing deciding whether a newly defined graph polynomial generalizes (or is generalized) by another one. Saeid Alikhani and Peng, Y.H. have introduced the domination polynomial of a graph.

Consider a graph G with vertex set V. A set $S \subseteq V$ is independent (or stable) if no two vertices in it are adjacent. The set $S \subseteq V$ is a *dominating set* if every vertex in $V - S$ is adjacent to a vertex of S. A subset S of V is an independent dominating set if it is independent and dominating in graph G. Independent

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domination number is the minimum size of an independent dominating set of G and is denoted by $\gamma_i(G)$.

The independent dominating polynomial of a graph G of order n is the polynomial $D_i(G,x) = \sum_{j=\gamma_i(G)}^{n} d_i(G,j)x^j$, where $d_i(G,j)$ is the number of independent dominating sets of G of size j.

The following previous results had been proven by Anwar Alwardi, P.M. Shivaswamy, and N.D. Soner [1]:

Theorem 1.1 *Let* $G \cong \bigcup_{j=1}^{n} G_j$. *Then* $D_i(G,x) = \prod_{j=1}^{n} D_i(G_j,x)$.

Theorem 1.2 *Let* G *be a complete graph* K_n *of* n *vertices. Then* $D_i(K_n,x) = nx$.

Theorem 1.3 *Let* G *be isomorphic to* $K_{m,n}$. *Then* $D_i(K_{m,n},x) = x^m(1+x^{n-m})$.

Theorem 1.4 *For any two graphs* G_1 *and* G_2, $D_i((G_1+G_2),x) = (D_i(G_1,x)) + (D_i(G_2,x))$.

Theorem 1.5 *Let* $G = F_m$ *be a friendship graph with* $2m+1$ *vertices. Then* $D_i(F_m,x) = x(1+2^m x^{m-1})$.

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2 RESULTS

The following results had been proven by the author:

Theorem 2.1 *Independent dominating polynomial of corona of complete graph* K_n *and any graph* G *is given by*

$$D_i(K_n \circ G, x) = nx(D_i(G,x))^{n-1} + (D_i(G,x))^n.$$

Proof: In each graph of the form $K_n \circ G$, we have two cases for an independent dominating set S.

Case 1. S contains u (the vertex of K_n) and $n-1$ arbitrary independent dominating sets of the copy G_v, where $v \neq u$. Note that vertex u dominates all the vertices of G_u. So, we can delete all vertices of the copy G_u and all vertices of K_n except u. Therefore, the arising graph is the disjoint union of $n-1$ copies of G and a vertex u. Since there are n vertices of K_n, by Theorem 1.1, the generating function for the number of independent dominating sets of the graph in this case is $nx(D_i(G,x))^{n-1}$.

Case 2. S does not contain any vertices of K_n and it is exactly an independent dominating set of G_v for all $v \in V(K_n)$. Note that every vertex of K_n is dominated by S. Hence, we can delete all vertices of K_n. Therefore, the resulting graph is

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the disjoint union of n copies of G. By Theorem 1.1, $(D_i(G, x))^n$ is the generating function in this case.

By addition principle, we have $D_i(K_n \circ G, x) = nx(D_i(G, x))^{n-1} + (D_i(G, x))^n$.

Using Theorems 2.1, 1.4, 1.2, 1.3 and 1.5, we have the following results.

Corollary 2.2

(i) $D_i(K_1 \circ H, x) = x + D_i(H, x)$.

(ii) $D_i(K_m \circ K_n, x) = mx(nx)^{m-1} + (nx)^m$.

(iii) $D_i(K_r \circ K_{m,n}, x) = rx(x^m + x^n)^{r-1} + (x^m + x^n)^r$.

(iv) $D_i(K_n \circ F_m, x) = nx(x + 2^m x^m)^{n-1} + (x + 2^m x^m)^n$.

Theorem 2.3 *Independent dominating polynomial of corona of complete bipartite graph $K_{m,n}$ and any graph G is given by*

$$D_i(K_{m,n} \circ G, x) = \sum_{s=1}^{m} x[D_i(G, x) + x]^{m-s}(D_i(G, x))^{n-1+s}$$
$$+ \sum_{t=1}^{n} x[D_i(G, x) + x]^{n-t}(D_i(G, x))^{m-1+t} + (D_i(G, x))^{m+n}.$$

Proof: Let $V_m = \{u_i : i = 1, 2, \ldots, m\}$ and $V_n = \{v_i : i = 1, 2, \ldots, n\}$ be the partite sets of $K_{m,n}$. In each graph of the form $K_{m,n} \circ G$, we have three cases for an independent dominating set S_i.

Case 1. S_i contains one vertex $u_i \in V_m$, an independent dominating set in $V(u_j + G_{u_j})$ for each u_j where $i \neq j$ and an independent dominating set in G_{v_k} for all $k = 1, 2, \ldots, n$. Note that if $u_i \in S_i$, then $u_{i-1} \notin S_i$ for all $i = 2, 3, \ldots, m$. Since vertex u_i dominates all the vertices of G_{u_i} and the vertices of V_n, we can delete all the vertices of the copy G_{u_i} and all vertices of V_n. Therefore, the arising graph is the disjoint union of $m - i$ copies of $G_{u_i} + u_i$, $n - 1 + i$ copies of G and a vertex u_i. By Theorems 1.1, 1.2 and 1.4, the generating function for the number of independent dominating sets corresponding to S_i is

$$\prod_{r=1}^{m-i} D_i(G_{u_r} + u_r, x) \prod_{k=1}^{n-1+i} D_i(G_{v_k}, x) \cdot D_i(K_1, x)$$
$$= [D_i(G, x) + x]^{m-i}(D_i(G, x))^{n-1+i} \cdot x$$
$$= x[D_i(G, x) + x]^{m-i}(D_i(G, x))^{n-1+i}$$

Since there are m such independent dominating sets, by addition principle, the generating function for the number of independent dominating sets of the graph in this case is

$$\sum_{s=1}^{m} x[D_i(G,x)+x]^{m-s}(D_i(G,x))^{n-1+s}.$$

Case 2. S_j contains one vertex $v_j \in V_n$, an independent dominating set in $V(v_i+G_{v_i})$ for each v_i where $i \neq j$ and an independent dominating set in G_{u_k} for all $k = 1, 2, \ldots, m$. This case is similar with case 1 where n and m are interchanged. Thus, the generating function for the number of independent dominating sets of the graph in this case is

$$\sum_{t=1}^{n} x[D_i(G,x)+x]^{n-t}(D_i(G,x))^{m-1+t}.$$

Case 3. S does not contain any vertices of $K_{m,n}$ and it only contains an independent dominating set in G_z for all $z \in V(K_{m,n})$. So, we can delete all vertices of $K_{m,n}$. Thus, the resulting graph is the disjoint union of $m+n$ copies of G. By Theorem 1.1, the generating function for the number of independent dominating sets of the graph in this case is $(D_i(G,x))^{m+n}$.

By addition principle, we have

$$D_i(K_{m,n} \circ G, x) = \sum_{s=1}^{m} x[D_i(G,x)+x]^{m-s}(D_i(G,x))^{n-1+s} + \sum_{t=1}^{n} x[D_i(G,x)+x]^{n-t}(D_i(G,x))^{m-1+t} + (D_i(G,x))^{m+n}.$$

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Using Theorems 2.3, 1.2, 1.3 and 1.5, we have the following results.

Corollary 2.4

(*i*) $D_i(K_{m,n} \circ K_g, x) = \sum_{s=1}^{m} x[(g+1)x]^{m-s}(gx)^{n-1+s} + \sum_{t=1}^{n} x[(g+1)x]^{n-t}(gx)^{m-1+t} + (gx)^{m+n}$.

(*ii*) $D_i(K_{m,n} \circ K_{g,h}, x) = \sum_{s=1}^{m} x(x^g+x^h+x)^{m-s}(x^g+x^h)^{n-1+s} + \sum_{t=1}^{n} x(x^g+x^h+x)^{n-t}(x^g+x^h)^{m-1+t} + (x^g+x^h)^{m+n}$.

(*iii*) $D_i(K_{m,n} \circ F_g, x) = \sum_{s=1}^{m} x(2x+2^g x^g)^{m-s}(x+2^g x^g)^{n-1+s} + \sum_{t=1}^{n} x(2x+2^g x^g)^{n-t}(x+2^g x^g)^{m-1+t} + (x+2^g x^g)^{m+n}$.

Theorem 2.5 *Independent dominating polynomial of corona of friendship graph F_m and any graph G is given by*

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$$D_i(F_m \circ G, x) = x(D_i(G,x))^{2m} + (D_i(K_2 \circ G, x))^m D_i(G,x).$$

Proof: In each graph of the form $F_m \circ G$, we have two cases for an independent dominating set S.

Case 1. S contains u (the common vertex of F_m) and $2m$ arbitrary independent dominating sets of the copy G_v, where $v \neq u$. Note that vertex u dominates all the vertices of F_m and the vertices of G_u. So, we can delete all vertices of the copy G_u and the all vertices of F_m except u. Thus, the arising graph is the disjoint union of $2m$ copies of G and a vertex u. Therefore, by Theorems 1.1 and 1.2, the generating function for the number of independent dominating sets of the graph in this case is $\prod_{r=1}^{2m} D_i(G,x) \cdot D_i(K_1,x) = (D_i(G,x))^{2m}(x) = x(D_i(G,x))^{2m}$.

Case 2. S does not contains u (the common vertex of F_m) and it only contains an independent dominating set in m copies of $K_2 \circ G$ and an independent dominating set in a copy of G_u. So, we can delete the vertex u of F_m. Thus, the resulting graph is the disjoint union of m copies of $K_2 \circ G$ and a copy of G. By Theorems 1.1 and 2.1, the generating function for the number of independent dominating sets of the graph in this case is

$$\begin{aligned}\prod_{r=1}^{m}(D_i(K_2 \circ G, x)) \cdot D_i(G,x) &= (D_i(K_2 \circ G, x))^m D_i(G,x) \\ &= [2x(D_i(G,x))^{2-1} + (D_i(G,x))^2]^m D_i(G,x) \\ &= [2x(D_i(G,x)) + (D_i(G,x))^2]^m D_i(G,x).\end{aligned}$$

By addition principle, we have $D_i(F_m \circ G, x) = x(D_i(G,x))^{2m} + [2x(D_i(G, x)) + (D_i(G,x))^2]^m D_i(G,x)$.

Using Theorems 2.5, 1.2, 1.3 and 1.5, we have the following results.

Corollary 2.6

(i) $D_i(F_m \circ K_n, x) = [n^{2m} + n(2n+n^2)^m]x^{2m+1}$.

(ii) $D_i(F_m \circ K_{g,h}, x) = x(x^g + x^h)^{2m} + [2x(x^g+x^h) + (x^g+x^h)^2]^m(x^g+x^h)$.

(iii) $D_i(F_m \circ F_n, x) = x(x+2^n x^n)^{2m} + [2x(x+2^n x^n) + (x+2^n x^n)^2]^m(x+2^n x^n)$.

3 RECOMMENDATIONS

In this paper the researcher considered the corona of some special graphs. The independent dominating polynomial of corona of other special graphs, composition of graphs and product of graphs are still to be determined.

Bibliography

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